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Hamiltonian and potentials in derivative pricing models: exact results and lattice simulations

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Abstract

The pricing of options, warrants and other derivative securities is one of the great success of financial economics. These financial products can be modeled and simulated using quantum mechanical instruments based on a Hamiltonian formulation. We show here some applications of these methods for various potentials, which we have simulated via lattice Langevin and Monte Carlo algorithms, to the pricing of options. We focus on *barrier* or *path dependent* options, showing in some detail the computational strategies involved.

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1. Introduction

Since Black and Scholes' (BS) option pricing model gained an almost immediate acceptance among the professional and academic communities, the trading of derivative securities skyrocketed. Derivative securities (such as *options*) are financial securities whose payoffs depends on other underlying securities, and the BS model was the first universally accepted modeling of these financial instruments.

However, in recent years, financial engineers¹ have created a variety of complex options that are collectively called *exotic options*.

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¹ Under this professional name perform actual research a large number of physicists and mathematicians, with a remarkable number of high energy physicists.

The payoffs on these options are considerably more diverse than the payoffs on standard BS options or on other straightforward generalizations of them. Some of these financial instruments, widely used in complex portfolios incorporating thousands of these (correlated) elementary instruments, may be easier to analyze using standard quantum mechanics.

While we focus on the possibility of applying fundamental theories to the real economy, we have to remark that the pattern seems to be much wider than expected, since even quantum field theory methods [1] have found their way to the financial world.

Most of the mathematical methods involved in the analysis of financial systems have been based, so far, on the simulation of stochastic processes by diffusion equations coupled to stochastic sources, i.e. stochastic equations of Langevin type. More recently, there has been an interest in the analysis of various financial instruments using the path integral formulation.

Use of a path integral formulation has some advantages. First, it is in close relation to the Lagrangian description of diffusion processes, second, it opens the way to the use of quantum mechanical methods, on which we briefly elaborate. After a description of the path integral in the BS model, which we study numerically, we turn our attention to the analysis of barrier options. Barrier options are studied here by simulating an artificial quantum mechanical model in which a potential $V(x)$ is added to the BS Lagrangian, as first suggested in Ref. [1]. Simulations are carried out using both Langevin and Monte Carlo methods, comparing in some limiting cases, where possible, our numerical results with analytical ones.

2. Langevin evolution

In the top down description of theoretical finance, a security $S(t)$ follows a random walk described by a Ito–Weiner process (or Langevin equation) as

$$\frac{dS(t)}{S(t)} = \phi dt + \sigma R(t) dt, \quad (1)$$

where $R(t)$ is a Gaussian white noise with zero mean and uncorrelated values at time t and t' ($\langle R(t)R(t') \rangle = \delta(t - t')$). ϕ is the drift term or expected return, while σ is a constant factor multiplying the random source $R(t)$, termed *volatility*.

As a consequence of Ito calculus, differentials of functions of random variables, say $f(S, t)$, do not satisfy Leibnitz's rule, and for a Ito–Weiner process with drift (1) one easily obtains for the time derivative of $f(S, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \phi S \frac{\partial f}{\partial S} + \sigma S \frac{\partial f}{\partial S} R. \quad (2)$$

The BS model is obtained by removing the randomness of the stochastic process shown above by introducing a random process correlated to (2). This operation, termed *hedging*, allows to remove the dependence on the white noise function $R(t)$, by constructing a *portfolio II*, whose evolution is given by the short-term risk free interest

rate r

$$\frac{d\Pi}{dt} = r\Pi . \tag{3}$$

A possibility is to choose $\Pi = f - (\partial f/\partial S)S$. This is a portfolio in which the investor holds an option f and short sells² an amount of the underlying security S proportional to $(\partial f/\partial S)$. A combination of (2) and (3) yields the BS equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} = rf . \tag{4}$$

There are some assumptions underlying this result. We have assumed absence of arbitrage, constant spot rate r , continuous rebalancing of the portfolio, no transaction costs and infinite divisibility of the stock.

The quantum mechanical version of this equation is obtained by a change of variable $S = e^x$, with x a real variable. This yields

$$\frac{\partial f}{\partial t} = H_{BS} f \tag{5}$$

with an Hamiltonian H_{BS} given by

$$H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right) \frac{\partial}{\partial x} + r . \tag{6}$$

Notice that one can introduce a quantum mechanical formalism and interpret the option price as a ket $|f\rangle$ in the basis of $|x\rangle$, the underlying security price. Using Dirac notation, we can formally reinterpret $f(x,t) = \langle x|f(t)\rangle$, as a projection of an abstract quantum state $|f(t)\rangle$ on the chosen basis.

In this notation, the evolution of the option price can be formally written as $|f,t\rangle = e^{tH}|f,0\rangle$, for an appropriate Hamiltonian H .

3. Options and barrier options

3.1. Generalities

Let the price at time t of a security be $S(t)$. A specific good can be traded at time t at the price $S(t)$ between a buyer and a seller. A forward contract is one where the seller (short position) agrees to sell the goods to the buyer (long position) at some time T in the future at a price $F(t,T)$ (the contract price). Notice that contract prices have a 2-time dependence (actual time t and maturity time T). Their difference $\tau = T - t$ is usually called *time to maturity*. Equivalently, the actual price of the contract is determined by the prevailing actual prices and interest rates and by the time to maturity.

² Short selling of the stock should be possible.

Entering into a forward contract requires no money, and the value of the contract for long position holders and short position holders at maturity T will be

$$(-1)^p(S(T) - F(t, T)), \quad (7)$$

where $p = 0$ for long positions and $p = 1$ for short positions. *Futures Contracts* are similar, except that after the contract is entered, any changes in the market value of the contract are settled by the parties. Hence, the cashflows occur all the way to expiry unlike in the case of the forward where only one cashflow occurs. They are also highly regulated and involve a third party (a *clearing house*). Forward, futures contracts and, as we will see, *options* go under the name of *derivative products*, since their contract price $F(t, T)$ depend on the value of the underlying security $S(T)$.

In the simplest option, such as a call option, we have seen that the payoff function is defined to be the value of the option at maturity time ($\tau = 0$). Therefore, the specific path followed by the underlying security is not relevant in order to establish the price at maturity, except for its final value.

Barrier options are, instead, path-dependent. This means that the payoff is dependent on the realized asset path, and certain aspects of the contract are triggered if the asset price, from start to end of the contract, becomes too high or too low.

Barrier options are very popular for various reasons. An investor may have very precise views about the behaviour of a security or he may use them for hedging specific cashflows, to decide to purchase them. In the following, when comparing path dependent options to the simplest options, such as standard calls or puts, we will refer to the latter as to *vanilla* options, using a common financial jargon.

3.2. Terminology and definitions

There are some advantages—and natural limitations—in purchasing a financial instrument such as a barrier option. If the purchaser wants the same payoff typical of a vanilla option, but believes that the upward movement of the underlying will not be likely, then he may decide to buy an *up-and-out* call option. The cost of this contract will be cheaper than the purchase of a corresponding plain vanilla option, but there will be severe limitations on the upward movement of the option.

The physical picture of an up-and-out option is that of a brownian motion of the underlying asset (x) that is immediately killed as soon as the asset hits (from below) the barrier $B(x = B)$, which is specified in the contract.

Similarly, a *down and out* provision renders the option worthless as soon as the asset price hits a barrier B from above. The payoffs in the two cases are given by

$$\begin{aligned} g_{UO}(x, K) &= \max(S_T - K)\theta(B - x), \\ g_{DO}(x, K) &= \max(S_T - K)\theta(x - B) \end{aligned} \quad (8)$$

for a up-and-out (UO) and a down-and-out (DO) option call respectively. Here, $\theta()$ denotes the standard step function. A terminology used to describe contracts with these features is *knocked out options*. In contracts of this type it is agreed there will not be any payoff if the barrier B is hit.

Similarly, the market offers contract with additional limitations on the allowed variation of the underlying asset. For instance, *double knock out* options have restrictions on the asset variability delimited by two barriers ($B_l < B_u$) both from above (B_u) and from below (B_l), and give zero payoff if any of the two barriers is hit by the asset from inception time t to expiry time T .

Knock in options are dual, in an obvious sense, to knock out options. *Knock in* options, in fact, are contracts that pay off as long as the barrier B is hit before expiry. If the barrier is hit, then the option is said to have *knocked in*, otherwise their payoff is null.

Furtherly categorizing these latter types of options, the position of the barrier respect to the initial value of the underlying allows to distinguish between *up-and-in* options and *down-and-in* options. The payoffs of these contracts are given by

$$\begin{aligned} g_{UI}(x, K) &= \max(S_T - K) \theta(x - B), \\ g_{DI}(x, K) &= \max(S_T - K) \theta(B - x). \end{aligned} \quad (9)$$

For definiteness, in the analysis that follows up, we will focus our attention to knocked out payoffs of the types described in Eq. (8).

In knocked out options, single or double, killing of the brownian motion is, needless to say, instantaneous, and takes place as soon as the brownian motion of the asset hits any of the barriers.

This aspect of the contract is an unpleasant feature since it introduces a discontinuity in the dynamics, with attached risk management problems both for option buyers and sellers. Such risks, for instance, are those due to erroneous price movements, or to an instantaneous spiky behaviour of an asset, moving upward or downward and penetrating a given barrier, which can lead an investor to the loss of all his investment. In other unpleasant situations, when large positions of options accumulate in the market and are all characterized by the same barrier, trading can drive the asset to the barrier, generating massive losses.

There are various ways by which more conservative and safer contracts can be defined, while maintaining some of the features of knock out options. This is achieved by introducing a finite knock out rate, thereby smoothing out the effect of the barrier. Our goal is to show how it can be implemented in a self-consistent path integral formulation and characterize the pricing of these path dependent options.

4. Quantum methods in finance

To establish a path integral description of a stochastic process we need a Lagrangian and the corresponding action. This can be easily worked out for the BS model, starting from the Hamiltonian given in Eq. (6). We easily get

$$L_{BS} = -\frac{1}{2\sigma^2} \left(\frac{dx}{dt} + r - \frac{1}{2}\sigma^2 \right)^2 - r \quad (10)$$

and the corresponding action, expressed in terms of time of maturity τ

$$S_{BS} = \int_0^\tau L_{BS}(t') dt' \tag{11}$$

which can be used to define a corresponding path integral for a fictitious quantum mechanical process in the variable x , the logarithm of the underlying asset

$$\langle x_f | e^{-\tau H_{BS}} | x_i \rangle = \prod_{t_i < t < t_f} \int_{-\infty}^{+\infty} dx(t) e^{S[x]} \tag{12}$$

with the boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$. The variable $x = \log(S)$ which identifies the quantum mechanical state of the system will be referred to as the stock price. The pricing kernel for the stock price is given by the

$$p_{BS}(x, x', \tau) = \int DX_{BS} e^{S_{BS}} = \langle x | e^{-\tau H_{BS}} | x' \rangle \tag{13}$$

with

$$\int DX_{BS} = \prod_{t=0}^\tau \int_{-\infty}^{+\infty} dx(t) . \tag{14}$$

4.1. Generalized potential

For certain options it is tempting [1] to introduce a potential $V(x)$ in order to set up a constraint on the stochastic process described by the stock price x .

The corresponding generalized Hamiltonian now reads

$$H_V = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} \sigma^2 - V(x) \right) \frac{\partial}{\partial x} + V(x) . \tag{15}$$

This Hamiltonian is equivalent to a stochastic process, as given in (86), with discounting done by $\exp(-\int dt V(x(t)))$.

It can be shown [1] that H_V obeys the martingale condition, and hence can be used for studying processes in finance.

The non-Hermiticity of H_V is of a particularly simple nature, and it can be shown [1] that for arbitrary V , H_V is equivalent by a similarity transformation to a Hermitian Hamiltonian H_{Eff} ³ given by

$$H_{\text{Eff}} = e^{-S} H_V e^S , \tag{16}$$

³Note that for more complex Hamiltonians such as the Merton–Garman H_{MG} finding the equivalent Hermitian Hamiltonian H_{Eff} is far from obvious.

where

$$H_{\text{Eff}} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\partial V}{\partial x} + \frac{1}{2\sigma^2} V^2 + \frac{1}{2} V + \frac{\sigma^2}{8} \tag{17}$$

and

$$s = \frac{1}{2}x - \frac{1}{\sigma^2} \int_0^x dy V(y) . \tag{18}$$

Note that H_{Eff} is Hermitian and hence its eigenfunctions form a complete basis; from this it follows that the Hamiltonian H_V can also be diagonalized using the eigenfunctions of H_{Eff} . In particular

$$H_{\text{Eff}}|\phi_n\rangle = E_n|\phi_n\rangle \tag{19}$$

$$\Rightarrow H_V|\psi_n\rangle = E_n|\psi_n\rangle , \tag{20}$$

where

$$|\psi_n\rangle = e^s|\phi_n\rangle , \tag{21}$$

$$\langle \tilde{\psi}_n | = e^{-s} \langle \phi_n | \neq \langle \psi_n | . \tag{22}$$

For the BS Hamiltonian H_{BS} we have $V(x) = r$ and hence

$$H_{BS} = e^s H_{\text{Eff}} e^{-s} \tag{23}$$

$$= e^{\alpha x} \left[-\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma \right] e^{-\alpha x} , \tag{24}$$

where

$$\gamma = \frac{(r + \sigma^2/2)^2}{2\sigma^2} , \quad \alpha = \frac{\sigma^2/2 - r}{\sigma^2} . \tag{25}$$

5. Path dependent options

We can have very complicated path dependent options since an option is an arbitrary random variable on the underlying sample space, or in other words, a completely arbitrary functional of the history of asset prices. For many but not all kinds of path dependent options, we can extend the technique of obtaining from the path integral a Hamiltonian for a quantity related to the option which is path independent and can therefore be represented as a wave function. The solution for the pricing kernel of this quantity then gives us the solution for the path dependent option. It must be noted that this quantity cannot be a traded asset as all traded assets evolve with the BS Hamiltonian

$$\hat{H}_{BS} = -rS \frac{\partial}{\partial S} - \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} \tag{26}$$

according to

$$\left(-\hat{H} + \frac{\partial}{\partial t}\right) f = 0 \tag{27}$$

in the BS model. Let us first look at how this is done for some relatively simple (but more complicated than simple barrier options) path dependent options.

5.1. Soft barrier options

These options have been considered in detail in Linetsky [2]. They are similar to the barrier options considered above but do not knock out the option completely when the barrier is hit. Instead, for soft barriers one discounts the final payoff, at some rate, by the exponential of the amount of time spent inside or outside the barrier. For example, for a down and out barrier is a discounted step option whose barrier is at B and strike price at K , the payoff at expiry is

$$e^{-V_{\tau_{B-}}}(S_T - K)_+ \tag{28}$$

where τ_{B-} is the time spent below the barrier B and V is the discounting factor. Considered as a path integral, the current price of the option is given by

$$\int dx' (e^{x'} - K)_+ \int_{x(0)=\ln S}^{x(T)=x'} \mathcal{D}x e^{S_{BS}} e^{-V_{\tau_{B-}}} . \tag{29}$$

Defining a potential

$$V(x) = \begin{cases} V, & x < \ln B, \\ 0, & x \geq \ln B \end{cases} \tag{30}$$

we see that the path integral is equivalent to

$$\int dx' (e^{x'} - K)_+ \int_{x(0)=\ln S}^{x(T)=x'} \mathcal{D}x e^S \tag{31}$$

with the action S now being given by

$$S = - \int dt L(x, \dot{x}) = - \frac{1}{2\sigma^2} \int dt \left(\left(\dot{x} - r - \frac{\sigma^2}{2} \right)^2 + r + V(x) \right) . \tag{32}$$

In other words, we have just introduced a potential into the problem. The Lagrangian is now

$$L = L_{BS} + V(x) \tag{33}$$

and the Hamiltonian now has an extra term $-V(x)$. The new Hamiltonian is therefore

$$\hat{H} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + r + V(x) . \tag{34}$$

Hence, the solution of the step option price in the BS model is equivalent to the solution of the plain vanilla option in a model with the above Hamiltonian.

More generally, the pricing of an option whose final payoff is

$$e^{-\int V(x(t)) dt} (S_T - K)_+ \tag{35}$$

is equivalent to the pricing of a plain vanilla call option in a model where the Hamiltonian is

$$\hat{H} = \hat{H}_{BS} + V(x) . \tag{36}$$

If we can find the eigenvalues and eigenfunctions of the operator \hat{H} , we can write down the pricing kernel using the decomposition

$$\langle x|e^{-\tau\hat{H}}|x'\rangle = \sum_n \langle x|n\rangle \langle n|e^{-\tau\hat{H}}|n\rangle \langle n|x'\rangle = \sum_n e^{-\tau E_n} \psi_n^*(x') \psi_n(x) , \tag{37}$$

where the eigenvalues of \hat{H} are E_n and the eigenfunctions corresponding to these eigenvalues are ψ_n . When the eigenvalues are continuous, the sum becomes an integral as is the case in the calculations for the single barrier options. In this case, we can consider the Laplace transform of the pricing kernel

$$\int_0^\infty d\tau e^{-s\tau} \langle x|e^{-\tau\hat{H}}|x'\rangle \tag{38}$$

which is seen to be the Green’s function of the operator $s + \hat{H}$ (this can also be directly seen from the Feynman representation of the pricing kernel). Once we find the Green’s function, we can perform the inverse Laplace transform to get the pricing kernel and hence the solution to the problem.

To see how this works, let us take the example of the simple barrier option. We modify the state space using the terms α and β so that we only deal with a standard Brownian motion. In that case, the potential only influences the boundary conditions, so we have to find the solution to the equation

$$\left(\frac{1}{2} \frac{d^2}{dx^2} - s \right) G(x, x'; s) = -\delta(x - x') \tag{39}$$

with the boundary conditions $G(x, x'; s) = 0, x, x' = b = \ln B$ and $\lim_{x \rightarrow \infty} G(x, x'; s) = 0$. The Green’s functions can be easily found using standard methods and the result is given by

$$G(x, x'; s) = \frac{2\sinh\sqrt{2s}(x - b)e^{-\sqrt{2s}(x' - b)}}{\sqrt{2s}} \Theta(x' - x) + \frac{2\sinh\sqrt{2s}(x - b)e^{-\sqrt{2s}(x - b)}}{\sqrt{2s}} \Theta(x - x') \tag{40}$$

whose inverse Laplace transform is the pricing kernel

$$p_{BS}(x, \tau; x') = \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{\tau\beta\sigma^2}{2} - \alpha(x - x')} \exp \left[-\frac{1}{2\tau\sigma^2} (x + x' - 2B)^2 \right] , \tag{41}$$

where p_{BS} is the BS pricing kernel and where the adjustment for the transformation has been made.

This technique is applied to find a closed form solution for the step option in Linetsky [2]. If we choose the variables such that the stock price $S = Be^{\sigma x}$ so that the barrier is at $x = 0$ and again only deal with the standard Brownian motion, we find that the Green’s function is given by

$$G(x, x'; s) = \begin{cases} \frac{1}{\sqrt{2S}} \left(e^{\sqrt{2s}|x-x'|} - \frac{\sqrt{s+V} - \sqrt{s}}{\sqrt{s+V} + \sqrt{s}} e^{\sqrt{2s}(x+x')} \right), & x, x' > 0, \\ \frac{e^{\sqrt{2(s+V)x} - \sqrt{2sx'}}}{\sqrt{2(s+V)} + \sqrt{2s}}, & x \leq 0, x' \geq 0, \\ \frac{e^{\sqrt{2(s+V)x'} - \sqrt{2sx}}}{\sqrt{2(s+V)} + \sqrt{2s}}, & x \geq 0, x' \leq 0, \\ \frac{1}{\sqrt{2(s+V)}} \left(e^{\sqrt{2(s+V)|x-x'|}} - \frac{\sqrt{s+V} - \sqrt{s}}{\sqrt{s+V} + \sqrt{s}} e^{\sqrt{2(s+V)(x+x')}} \right), & x, x' < 0 \end{cases} \tag{42}$$

whose inverse Laplace transform gives us the pricing kernel.

5.2. Asian options

There are several options which cannot be put into a simple form by discounting alone. One such option which is also fairly popular in the market is the Asian option which has been considered in detail in the literature. The Laplace transform of an out of the money Asian option was found in Geman and Yor [3]. The payoff of the Asian option is defined to be

$$\max \left(0, \frac{1}{T} \int_0^T S(t) dt - K \right). \tag{43}$$

We can write the option price as a path integral

$$\frac{1}{2\pi} \int dp \int dv \int \mathcal{D}x e^{S_{BS}} e^{ip((1/T) \int_0^T dt e^{x(t)-v})} (v - K)_+ \tag{44}$$

using a standard expression for the Dirac delta function. We see that we can consider this expression as a plain vanilla call option with a Lagrangian modified by $-(ip/T) e^x$. If it were not for the term i , this would be a reducible to a relatively standard problem. However, since the additional potential term is now complex, the solution is not so simple.

One option which is a fairly good approximation for the Asian option but which is easily solvable is the geometric Asian option. Its final payoff is defined

to be

$$\max(0, e^{(1/T) \int_0^T x(t) dt} - K) . \tag{45}$$

To solve this, let us write the BS evolution as

$$\frac{dx}{dt} = \left(r - \frac{\sigma^2}{2} \right) + \sigma \eta(t) , \tag{46}$$

where $\eta(t)$ is white noise. Hence,

$$x(t) = x(0) + \left(r - \frac{\sigma^2}{2} \right) t + \int_0^t dt' \eta(t') \tag{47}$$

and

$$\frac{1}{T} \int_0^T dt x(t) = x(0) + \frac{1}{2} \left(r - \frac{\sigma^2}{2} \right) t + \frac{\sigma}{T} \int_0^T (T - t) \eta(t) dt . \tag{48}$$

To find the distribution of the last term, we make use of the generating function for white noise to give

$$\frac{1}{2\pi} \int \mathcal{D}\eta e^{-\frac{1}{2} \int_0^T dt \eta^2(t)} e^{ip(\sigma/T) \int_0^T dt (T-t) \eta(t) - v} \tag{49}$$

which evaluates to

$$\sqrt{\frac{3}{2\pi\sigma^2 T}} e^{-3v^2/2\sigma^2 T} . \tag{50}$$

In other words, the distribution of $(1/T) \int_0^T x(t) dt$ is $N(0, \sigma^2 T/3)$. Therefore, the price of the geometric Asian option is given by

$$c = SN(d_1) - Ke^{-r(T-t)} N(d_2) , \tag{51}$$

where

$$d_1 = \frac{\sqrt{3}(\ln(S/K) + (r/2 + \sigma^2/12)(T - t))}{\sigma\sqrt{T - t}} ,$$

$$d_2 = \frac{\sqrt{3}(\ln(S/K) + (r/2 - \sigma^2/4)(T - t))}{\sigma\sqrt{T - t}} . \tag{52}$$

5.3. Seasoned options

In the above discussion, we have only discussed how to value path dependent options where the path dependence starts at the present. In practice, the problem of solving for the price of path dependent options after they have been initiated is very important. Such options are called seasoned options. In many cases, the valuation of seasoned options proceeds very similarly to that of new options.

Let us consider a seasoned soft barrier option with discounting by a potential $V(x)$ at time $t > 0$ where the path dependence has started at zero time. We denote the maturity

time be T . The value of this option is given by

$$e^{-\int_0^T dt' V(x(t'))} \int \mathcal{D}x e^{S_{BS}} e^{-\int_t^T dt' V(x(t'))} (e^{x(T)} - K)_+, \tag{53}$$

where we should take into account the discounting until time t separately since the history is already known. We see that apart from this factor, there is no substantial difference between the valuation of the new and seasoned options.

For Asian options, we see that we can value the seasoned option provided we can value the new option since the probability distribution of the remaining part of the average will determine the option price at the time when combined with information about the contribution to the average of the revealed historical price. Hence, we see that we can price seasoned Asian options if we can price new Asian options.

6. Solving the double knock out barrier option

A double barrier option is an option whose value reduces to zero whenever the price of the underlying instrument hits the barriers which we denote by e^a and e^b . Hence, the price of a double knock out barrier European call option expiring at time T and with strike price K at time t_0 provided it has not already been knocked out will be given by

$$e^{-r(T-t_0)} E_t[(e^{x(T)} - K)_+] \mathbf{1}_{a < x(t) < b, t_0 < t' < T}, \tag{54}$$

where $\mathbf{1}$ stands for the indicator function. It is sufficient to solve for the probability distribution of $x(T)$ for those paths which do not go outside the barriers (in other words, the pricing kernel).

Written as a path integral, the formula is

$$e^{-r(T-t_0)} \int \mathcal{D}x \Theta(x(t) - a) \Theta(b - x(t)) e^{S_{BS}(x(t))} (e^{x(T)} - K)_+, \tag{55}$$

where S_{BS} is the BS action

$$S_{BS} = -\frac{1}{2\sigma^2} \int dt \left(\dot{x} + r - \frac{\sigma^2}{2} \right)^2. \tag{56}$$

While the step functions look complicated in the path integral, they can be seen to be having the effect of an infinite potential barrier since they effectively prohibit the path from entering the forbidden region outside the barriers. Hence, the problem might be better solved using the Hamiltonian and this is indeed the case.

In the Schrödinger formulation, the above problem is to find the pricing kernel for a system with the Hamiltonian

$$\hat{H} = \hat{H}_{BS} + V(x), \tag{57}$$

where the BS Hamiltonian is given by

$$\hat{H}_{BS} = -\frac{\sigma^2}{2} \frac{\partial}{\partial x^2} + \left(\frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} \tag{58}$$

and the potential $V(x)$ is given by

$$V(x) = \begin{cases} \infty, & x < a, \\ 0, & a < x < b, \\ \infty, & x > b. \end{cases} \tag{59}$$

This is very similar to the well known problem of a particle in an infinite potential well except that the Hamiltonian has an extra term involving $\partial/\partial x$ which makes it non-Hermitian.

This problem can be solved by transforming the underlying wave functions. By making the transformation $\langle x|\phi\rangle = e^{-\alpha(x-a)}\langle x|\psi\rangle$ and $\langle\phi|x\rangle = e^{-\alpha(x-a)}\langle\psi|x\rangle$, where $|\phi\rangle$ are the vectors in the new (Hilbert) space, $|\psi\rangle$ and $\langle\psi|$ are the original vectors and their duals respectively and $\alpha = (\sigma^2/(2-r))/\sigma^2$. In this new space, the BS Hamiltonian takes the simple Hermitian form $-(\sigma^2/2)\partial^2/\partial x^2$.

The problem is now identical to that of a quantum mechanical particle of mass $1/\sigma^2$ (in units where $\hbar = 1$) in an infinite potential well. As is well known in this case, the allowed momenta are $p_n = n\pi/(b-a)$. The eigenfunctions are hence given by

$$\langle x|\psi_n\rangle = e^{\alpha(x-a)}\langle x|\phi_n\rangle = \sqrt{\frac{2}{b-a}}ie^{\alpha(x-a)}\sin p_n(x-a), \tag{60}$$

$$\langle\tilde{\psi}_n|x\rangle = e^{-\alpha(x-a)}\langle\phi_n|x\rangle = -\sqrt{\frac{2}{b-a}}ie^{-\alpha(x-a)}\sin p_n(x-a), \tag{61}$$

where $\langle x|\phi_n\rangle$ are the eigenfunctions of the quantum mechanical particle in an infinite potential well.

The eigenfunctions are orthonormal and form a complete basis since

$$\begin{aligned} \sum_{n=1}^{\infty} \langle x|\psi_n\rangle\langle\tilde{\psi}_n|x'\rangle &= \frac{2}{b-a}e^{\alpha(x-x')} \sum_{n=1}^{\infty} \sin p_n(x-a)\sin p_n(x'-a) \\ &= \frac{1}{2(b-a)}e^{\alpha(x-x')} \sum_{n=-\infty}^{\infty} \left(\exp \frac{i n \pi}{b-a}(x-x') \right. \\ &\quad \left. - \exp \frac{i n \pi}{b-a}(x+x'-2a) \right) \\ &= \frac{\pi}{b-a}e^{\alpha(x-x')} \left(\delta \left(\frac{\pi(x-x')}{b-a} \right) - \delta \left(\frac{\pi(x+x'-2a)}{b-a} \right) \right) \\ &= \delta(x-x') \end{aligned} \tag{62}$$

since $a < x < b$ and $a < x' < b$.

The pricing kernel is hence given by

$$\begin{aligned}
 \langle x | e^{-\tau \hat{H}} | x' \rangle &= \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \langle x | \psi_n \rangle \langle \tilde{\psi}_n | e^{-\tau \hat{H}} | \psi_{n'} \rangle \langle \tilde{\psi}_{n'} | x' \rangle \\
 &= \sum_{n=1}^{\infty} \langle x | \psi_n \rangle \langle \tilde{\psi}_n | x' \rangle e^{-\tau E_n} \\
 &= \frac{1}{2(b-a)} \exp\left(-\frac{\tau \sigma^2 \beta}{2} + \alpha(x-x')\right) \\
 &\quad \times \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\tau \sigma^2 p_n^2}{2}\right) (e^{i p_n(x-x')} - e^{i p_n(x+x'-2a)}) \\
 &= \frac{1}{2(b-a)} \exp\left(-\frac{\tau \sigma^2 \beta}{2} + \alpha(x-x')\right) \sum_{n=-\infty}^{\infty} \int dy \delta(y-n) \\
 &\quad \times \exp\left(-\frac{y^2 \pi^2 \tau \sigma^2}{2(b-a)^2}\right) \left(\exp\frac{iy\pi(x-x')}{b-a} - \exp\frac{iy\pi(x+x'-2a)}{b-a}\right) \\
 &= \sqrt{\frac{1}{2\pi\tau\sigma^2}} \exp\left(-\frac{\tau \sigma^2 \beta}{2} + \alpha(x-x')\right) \\
 &\quad \times \sum_{n=-\infty}^{\infty} \left(\exp -\frac{(x-x'+2n(b-a))^2}{2\tau\sigma^2}\right. \\
 &\quad \left. - \exp -\frac{(x+x'-2a-2n(b-a))^2}{2\tau\sigma^2}\right), \tag{63}
 \end{aligned}$$

where

$$\beta = \frac{(\sigma^2/2 + r)^2}{\sigma^4} \tag{64}$$

and the identity

$$\delta(y-n) = \sum_{n=-\infty}^{\infty} e^{2\pi i n y} \tag{65}$$

has been used.

Hence, we see that the pricing kernel (apart from the drift terms) is given by an infinite sum of Gaussians. To check its reasonableness, we check the value in the limits $b \rightarrow \infty$ and $a \rightarrow -\infty$. In the former case, only the $n = 0$ term contributes and in the latter, only the $n = 0$ and $n = 1$ terms contribute. It is easy to see that, in both cases, the result reduces to the solution for the single knockout barrier pricing kernel. When both limits are simultaneously active, only the first term in the $n = 0$ term exists and it is easily seen that gives rise to the well known BS pricing kernel.

We can now evaluate the price of a double barrier European call option using the pricing kernel from (63). The result is seen to be

$$f = \sum_{n=-\infty}^{\infty} \left(e^{-2nz(b-a)} e^{2n(b-a)} SN(d_{n1}) - Ke^{-r\tau} N(d_{n2}) \right) - S^{2\alpha} e^{-2\alpha n(b-a)-a} \left(e^{2n(b-a)} \frac{e^{2a}}{S} N(d_{n3}) - Ke^{-r\tau} N(d_{n4}) \right), \tag{66}$$

where

$$d_{n1} = \frac{\ln(S/K) + 2n(b-a) + \tau(r + \sigma^2/2)}{\sigma\sqrt{\tau}}, \tag{67}$$

$$d_{n2} = \frac{\ln(S/K) + 2n(b-a) + \tau(r - \sigma^2/2)}{\sigma\sqrt{\tau}} = d_{n1} - \sigma\sqrt{\tau}, \tag{68}$$

$$d_{n3} = \frac{\ln(e^{2a}/SK) + 2n(b-a) + \tau(r + \sigma^2/2)}{\sigma\sqrt{\tau}}, \tag{69}$$

$$d_{n4} = \frac{\ln(e^{2a}/SK) + 2n(b-a) + \tau(r - \sigma^2/2)}{\sigma\sqrt{\tau}} = d_{n3} - \sigma\sqrt{\tau}. \tag{70}$$

7. Monte Carlo simulations

The pricing kernel is the fundamental quantity to compute using the functional integral. Related attempts can be found in the literature [4]. We assume a discretization of the time to maturity τ in intervals $\varepsilon = \tau/N$, with N an arbitrary (large) integer.

For instance, for the BS model one gets the action

$$S_{BS} = \varepsilon \sum_{i=1}^N L_{BS}(i) \tag{71}$$

with

$$L_{BS}(i) = -\frac{1}{2\sigma^2} \left(\frac{x_i - x_{i-1}}{\varepsilon} + r - \frac{\sigma^2}{2} \right)^2, \tag{72}$$

where we have introduced discretized positions (x_i) for the variable.

For this purposes, we have used a standard Metropolis algorithm. If thermalization is slow, it is possible to resort to use sequentially Metropolis updates and cluster updates. The latter is an update for the embedded Ising dynamics in the lattice variables $x_i/|x_i|$ (Swendsen–Wang, Wolff), and is included in for a faster generation of the thermalized paths of the stock price $x(t)$.

For processes involving a stochastic volatility ($y = \log(V)$) the expression of the path integral is more complicated and can be found in [5]. From now on we will just consider the case of a constant volatility.

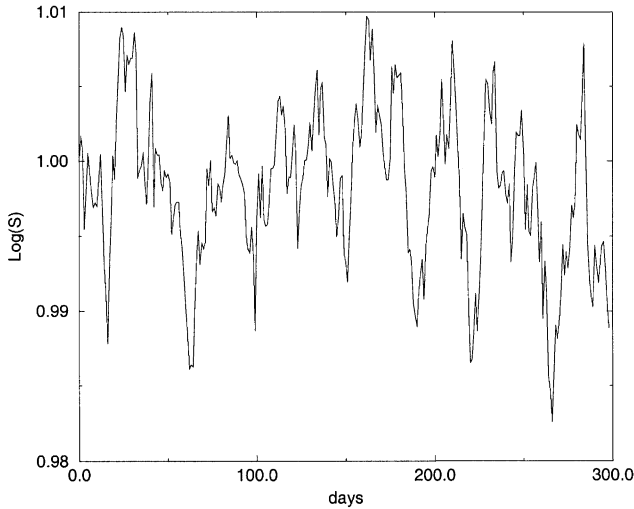


Fig. 1. An example of thermalized path obtained from the simulation of the path integral (BS) with $r = 0.05$ and $\sigma = 0.12$.

If we denote by $g(x, K)$ the payoff function, with a strike price K , in this case the value of the option (its price) is given by the Feynman–Kac formula

$$f(t, x) = \int_{-\infty}^{\infty} dx \langle x | e^{-(T-t)H_{BS}} | x' \rangle g(x', K). \quad (73)$$

In actual simulations, it is convenient to compute directly the option price rather than the propagator itself. The simulation is done by taking the initial point x fixed, and letting the final point evolve according to its quantum dynamics. In this way a path (x, x') is generated. After the first thermalization, x' is allowed to undergo quantum fluctuations, at fixed x . Each x' is then convoluted with the payoff function and an average is performed. Finally, this procedure is repeated for several x values, so to obtain the option price at time to maturity τ .

Figs. 1–3, illustrate some simple results obtained by the Monte Carlo method. For illustrative purposes, we show the behaviour of the BS model. Fig. 1 shows a typical thermalized path, generated from a given initial value x (at current time $t = \tau$) assuming a maturity of 300 days, while in Fig. 2 we have plotted several path for different starting values x of the stock at current time τ . We have chosen an interest rate $r = 0.05$ and a 12 percent volatility σ . Finally, in Fig. 3 we compare the analytical and the numerical evaluation of the BS option price with a low resolution for (73), in order to separate the two curves, which otherwise would overlap completely, in order to illustrate the convergence of the Metropolis algorithm.

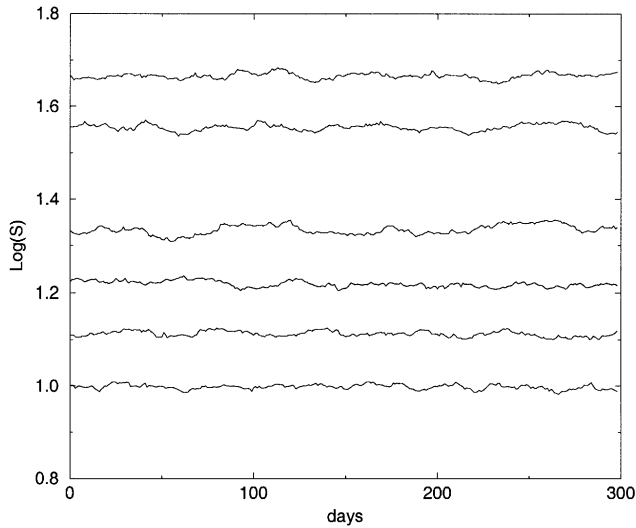


Fig. 2. Several thermalized paths for (BS) with $r = 0.05$ and $\sigma = 0.12$.

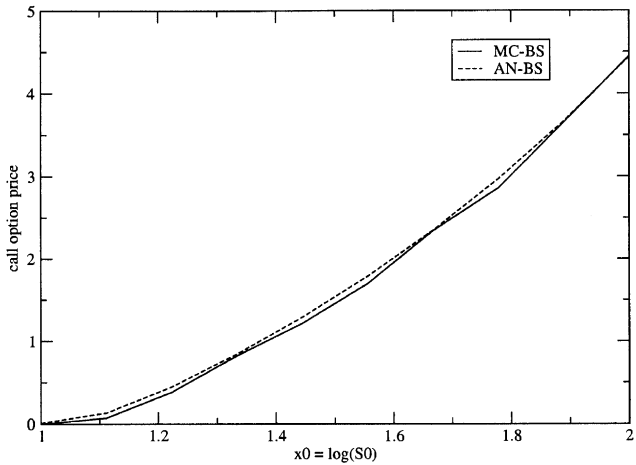


Fig. 3. Call option price for strike price 3 versus the logarithm of the initial value of the stock $x_0 = \log(S_0)$. The parameters are fixed as in Fig. 1. Shown is the analytical result versus the Monte Carlo result, with a low resolution of 10,000 configurations.

8. Langevin calculation of option prices with potential: numerical methods

As is well known from quantum mechanics, the path integral

$$\int \mathcal{D}x e^{(i/\hbar) \int L(x, \dot{x}) dt} \tag{74}$$

with the Lagrangian L given by

$$\frac{1}{2}m\dot{x}^2 - V(x) \tag{75}$$

is the evolution kernel for a one-dimensional system of a particle of mass m moving in a potential $V(x)$. This is very similar to the path integral for standard Brownian motion $W(t)$ given by

$$\int \mathcal{D}W e^{-\frac{1}{2} \int \dot{W}^2(t) dt} . \tag{76}$$

The path integral for a more general stochastic process described by the stochastic differential equation (in physics, the equation is usually written out with everything divided by dt and with $dW(t)/dt$ replaced by η which represents white noise and the equation is then called the Langevin equation)

$$dx(t) = a(x) dt + \sigma(x) dW(t) \tag{77}$$

can be found by solving the above for \dot{W} and using the path integral for white noise (76). One easily obtains the path integral

$$\int \mathcal{D}x e^{-\int dt \frac{1}{2\sigma^2(x(t))} (\dot{x} - a(x(t)))^2} . \tag{78}$$

When we perform a Wick rotation for the action in (74), we get the Lagrangian for Brownian motion if $V(x) = 0$. Hence, a free quantum mechanical particle can in some sense be considered to be undergoing Brownian motion and can be modelled by the stochastic differential equation

$$dx(t) = \sqrt{\frac{\hbar}{m}} dW(t) . \tag{79}$$

The path integral for the Euclidean action can be numerically simulated using the Metropolis algorithm. Another method is to use the analogy and directly integrate the stochastic differential equation (79). The latter is more efficient as one does not have the problem of correlation between successive configurations which reduces the accuracy of Monte Carlo calculations. There does seem to be a problem in that potentials cannot be included. However, this can be handled by including a killing term in the stochastic differential equation whose connection with the potential becomes clear when we consider the Hamiltonian. The integration of stochastic differential equations can also give efficient numerical calculations in quantum field theory, especially for gauge invariant theories where gauge fixing is trivial in this framework. This is dealt with in great detail in Namiki [6].

In the case of stock option pricing, we are usually dealing with the BS stochastic differential equation

$$dS(t) = rS(t) dt + \sigma S(t) dW(t) \tag{80}$$

whose Hamiltonian is given by

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} \tag{81}$$

and Lagrangian by

$$- \frac{1}{2\sigma^2 S^2} (\dot{S} - rS)^2 . \tag{82}$$

This becomes much simpler when we transform the stochastic differential equation to the variable $x = \ln S$ using Itô’s lemma to give

$$dx(t) = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \tag{83}$$

with the Hamiltonian

$$\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial}{\partial x} \tag{84}$$

and much simpler Lagrangian

$$- \frac{1}{2\sigma^2} \left(\dot{x} - r + \frac{\sigma^2}{2} \right)^2 . \tag{85}$$

We will consider a somewhat more general stochastic process for the BS stochastic differential equation given by

$$dx(t) = \left(V(x) - \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \tag{86}$$

which represents a situation where r is given by $V(x)$. We can recover the BS result by setting $V(x) = r$ and a general potential $V(x)$ is interesting as a mathematical exercise. It must be noted that this potential is very different from the potential used for the barrier options in Section 5.1 (for e.g., in Eq. (28)).

To accommodate the discounting of all assets by the money market account, we have to include a killing term in the stochastic differential equation so that all expectations are discounted by $\exp(-\int dt V(x(t)))$. This changes the Hamiltonian to

$$\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \left(V(x) - \frac{\sigma^2}{2} \right) \frac{\partial}{\partial x} - V(x) . \tag{87}$$

The reason the Hamiltonian above differs by an overall sign from the BS Hamiltonian is because in pricing of options, one is considering the *backward* Fokker–Planck equation that results from the stochastic Langevin equation.

The Hamiltonian above can be simulated by numerically integrating the BS stochastic differential equation with r replaced by $V(x)$ and putting in the discount factor explicitly when calculating expectations.

Alternatively, general barrier options can be considered by keeping the BS process for x but introducing an extra killing term $V(x)$. In that case, we get the Hamiltonian

$$\hat{H} = - \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + r + V(x) , \tag{88}$$

where the killing term r is included because even the plain vanilla option price must be discounted by the money market account to get its current price. This can also be simulated by numerically integrating the BS stochastic differential equation and putting in the discount factor explicitly when calculating expectations.

In general, therefore, option pricing with Hamiltonian (88) is equivalent to solving the Itô stochastic differential equation

$$dx = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dW(t), \tag{89}$$

where $W(t)$ is a standard Wiener process and then taking all expectations discounted by $\exp(-\int (r + V(x(t))) dt)$ [7]. If we are using the Hamiltonian

$$\hat{H} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - V(x) \right) \frac{\partial}{\partial x} + V(x) \tag{90}$$

this becomes equivalent to solving the stochastic differential equation

$$dx = \left(V(x) - \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \tag{91}$$

with discounting now being done by the factor $\exp(-\int V(x(t)) dt)$.

This can be done in a straightforward manner numerically. The stochastic differential equation can be solved by the Euler method to give sample paths $x(t_i), 0 \leq i \leq N$ where $t_0 = 0, t_i = iT/N = i\varepsilon$ and the discounting of the final value of the option can be done with the factor given above. More explicitly, the simulation is done by numerically integrating the Itô stochastic differential equation (89) using

$$x_{i+1} = x_i + \left(r - \frac{\sigma^2}{2} \right) (\varepsilon) + \sigma\sqrt{\varepsilon}Z \tag{92}$$

or (91) using

$$x_{i+1} = x_i + \left(V(x_i) - \frac{\sigma^2}{2} \right) (\varepsilon) + \sigma\sqrt{\varepsilon}Z, \tag{93}$$

where x_i is $x(t_i)$ and Z is a normally distributed random numbers with unit variance. Such random numbers can be obtained from the usual uniformly distributed random numbers in the range [0,1) by the transformation

$$\zeta_1 = \sqrt{-2 \ln \xi_1} \cos 2\pi\xi_2, \tag{94}$$

$$\zeta_2 = \sqrt{-2 \ln \xi_1} \sin 2\pi\xi_2, \tag{95}$$

where ξ_1 and ξ_2 are distributed uniformly on [0,1) and ζ_1 and ζ_2 are now normally distributed with unit variance.

The modified call option price described by (88) is then given by

$$E \left[\exp \left(-rT - \varepsilon \sum_{i=0}^{N-1} V(x_i) \right) (e^{x_N} - K)_+ \right], \tag{96}$$

where the x are generated using (92). The modified call option prices described by (90) are given by

$$E \left[\exp \left(-\varepsilon \sum_{i=0}^{N-1} V(x_i) \right) (e^{x_N} - K)_+ \right], \tag{97}$$

where the x are generated using (93).

9. Results of numerical simulations

The initial stock price S_0 is assumed to be 100. We make use of the variable $x = \ln S$ as explained above and we set $x_0 = \ln S_0$.

There are two ways we use the potential as explained in the previous section. In the first case, the killing term (coefficient of constant) in the Hamiltonian is $r + V(x)$ while the drift term, the coefficient of $\partial/\partial x$, is still $r - \sigma^2/2$. This corresponds to a constant interest rate r and where the option payoff at expiry is defined in a path dependent way as

$$g[S] = \left(\exp \left(- \int_0^T V(S(t)) dt \right) S(T) - K \right)_+, \tag{98}$$

where $(y)_+$ stands for $\max(0, y)$. In this case, the potential V and r are separated as they have different interpretations.

In the second case, the interest rate is assumed to be given as a function of the underlying security price and is equal to $V(x)$. In this case, the Hamiltonian has V in both the drift and killing (coefficient of constant) terms. The option payoff $g(S)$ is then *path-independent* and is the usual call option payoff $\max(0, S - K)$.

Figs. 4–7 refer to potential as used in the first sense. The interest rate was fixed at 5%, the volatility σ^2 at 0.25/year and the time to expiry to one year. In Fig. 4, we show a plot of the call option price versus the strike price with

$$V(x) = \begin{cases} 1, & x \leq x_0, \\ 0, & x > x_0. \end{cases} \tag{99}$$

In Fig. 5, the potential used is

$$V(x) = \begin{cases} 0, & x \leq x_0, \\ 1, & x > x_0. \end{cases} \tag{100}$$

while in Fig. 6, it is

$$V(x) = \begin{cases} -1, & x \leq x_0, \\ 0, & x > x_0. \end{cases} \tag{101}$$

and in Fig. 7, it is

$$V(x) = \begin{cases} 0, & x \leq x_0, \\ -1, & x > x_0. \end{cases} \tag{102}$$

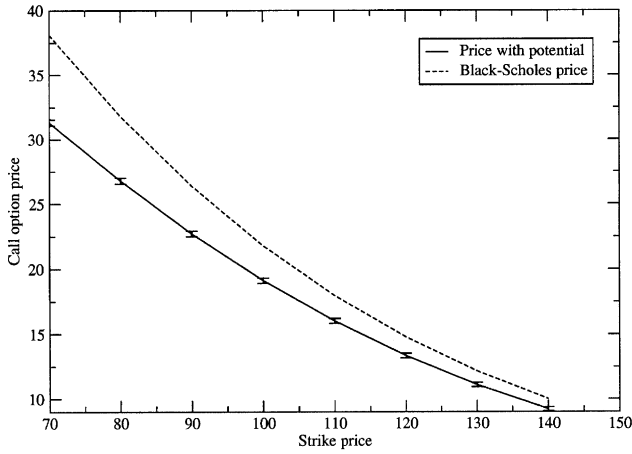


Fig. 4. Plot of call option price against strike price for a purely discounting potential of form (99) and for the BS case.

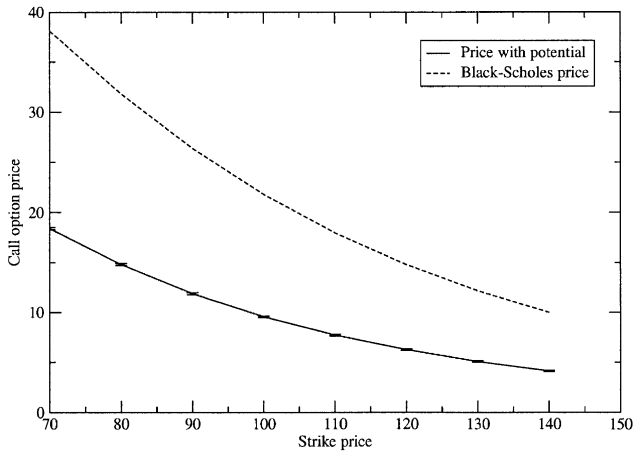


Fig. 5. Plot of call option price against strike price for a purely discounting potential of form (100) and for the BS case.

Figs. 8–11 refer to the potential as used in the second sense. The volatility at 0.25/year and the time to expiry at one year. In Fig. 8, we show a plot of the call option price versus the strike price with

$$V(x) = \begin{cases} 1.05, & x \leq x_0, \\ 0.05, & x > x_0. \end{cases} \tag{103}$$

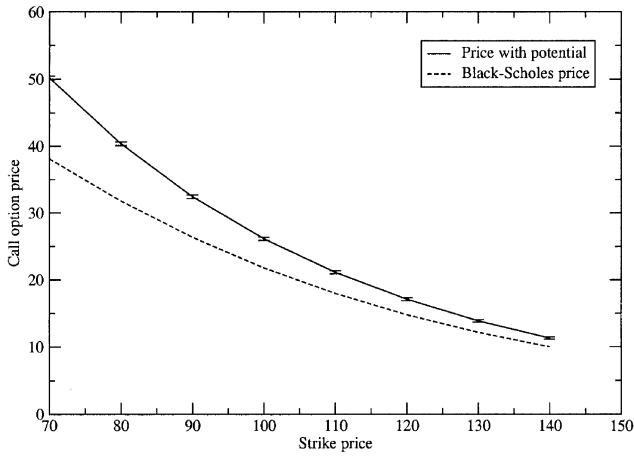


Fig. 6. Plot of call option price against strike price for a purely discounting potential of form (101) and for the BS case.

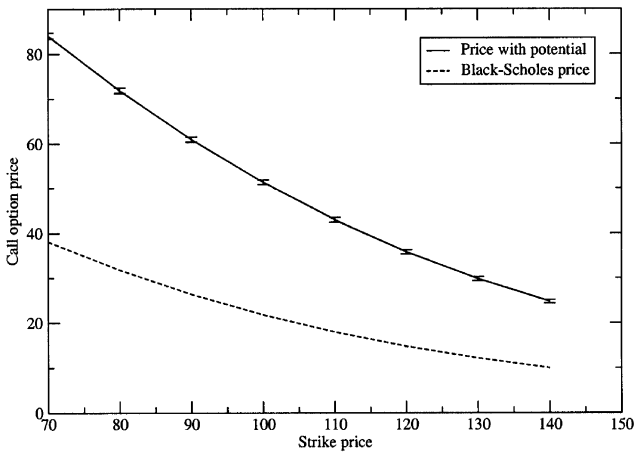


Fig. 7. Plot of call option price against strike price for a purely discounting potential of form (102) and for the BS case.

In Fig. 9, the potential used is

$$V(x) = \begin{cases} 0.05, & x \leq x_0, \\ 1.05, & x > x_0. \end{cases} \quad (104)$$

while in Fig. 10, it is

$$V(x) = \begin{cases} -0.95, & x \leq x_0, \\ 0.05, & x > x_0. \end{cases} \quad (105)$$

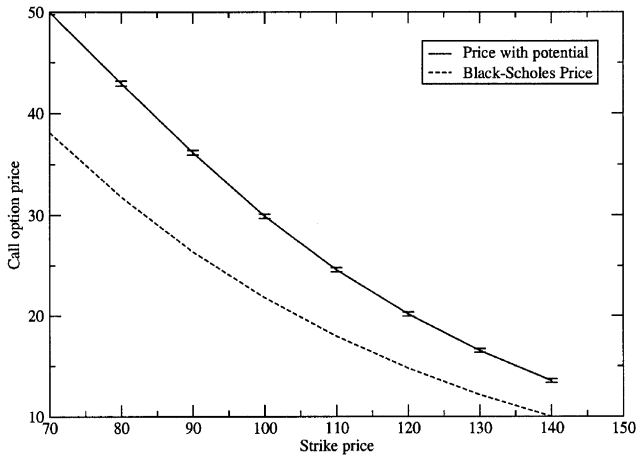


Fig. 8. Plot of call option price against strike price for a potential of form (103) and for the BS case.

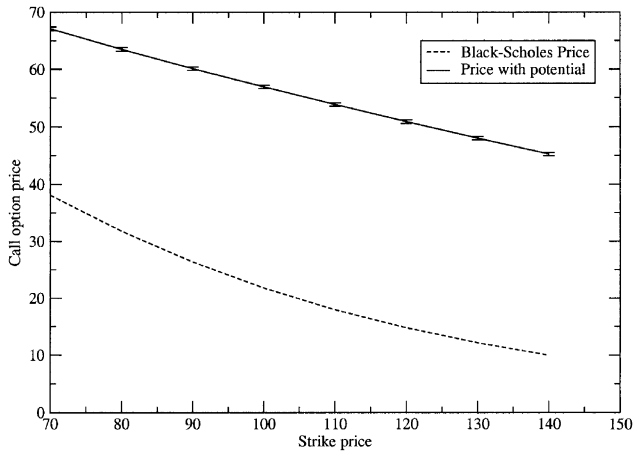


Fig. 9. Plot of call option price against strike price for a potential of form (104) and for the BS case.

and in Fig. 11, it is

$$V(x) = \begin{cases} 0.05, & x \leq x_0, \\ -0.95, & x > x_0. \end{cases} \tag{106}$$

The last two cases are unrealistic in practice as interest rates can never go negative.

By a simple analysis of the stochastic processes involved, we can understand why the prices of the options have the forms shown in the graphs. For example, in Figs. 4 and 5 corresponding to the prices of options with a simple discounting potentials of form (99) and (100) respectively, it is obvious from the discounting that the price of

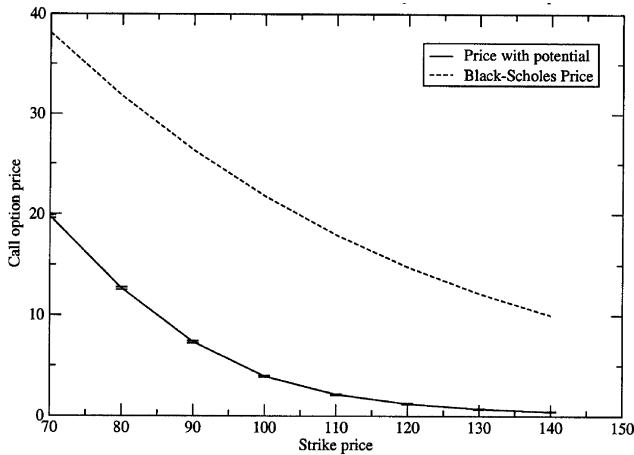


Fig. 10. Plot of call option price against strike price for a potential of form (105) and for the BS case.

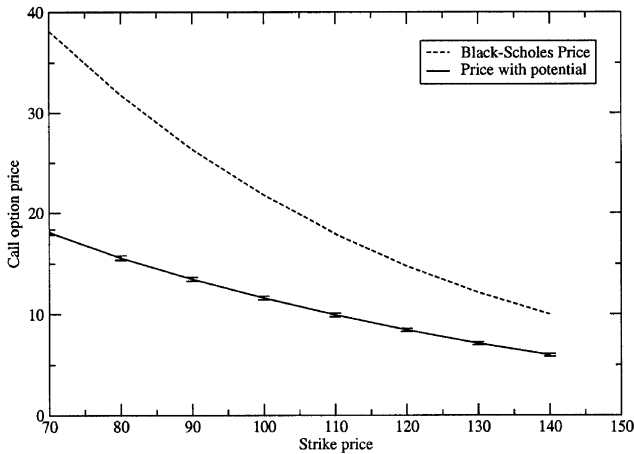


Fig. 11. Plot of call option price against strike price for a potential of form (106) and for the BS case.

the barrier options will be lower than that of the vanilla options. It can also be seen that the price for the option with potential (100) will be lower than that of one with potential (99) since an option which expires in the money is more likely to have spent time above rather than below the barrier. The graphs in Figs. 6 and 7 can also be understood similarly. When the potential is used in the second sense of the term, the analysis is somewhat more subtle. In this case, there are two competing effects, the risk-neutral measure tending to increase the price with the potential and the discounting tending to decrease it. From the graphs in Figs. 8–11, we see that the first effect wins out since the prices are higher with higher potentials (for example, contrast graph 8

using potential (103) and graph 10 using potential (105)). The difference between the results of using potential (103) and (104) or (105) and (106) can again be understood to be due to the increased likelihood of options expiring in the money (which are the only ones that contribute to the price) in the ensemble being more likely to be in the region above the barrier.

10. Conclusions

The path integral formulation of financial instruments, as shown in this work, is a promising approach to the pricing to derivative products which shows a remarkable flexibility. We have presented several applications of the method and have provided a general strategy-based on the use of a potential in the modeling of barrier options to analyze these instruments.

Specifically, we have compared Langevin simulations and Monte Carlo simulations and shown that a discount on the price of these options has indeed a rather simple interpretation in terms of paths of the underlying security. We believe that these strategies will turn out to be very effective for the simulation of complex portfolios, as well as for the inclusion of constraints in the evolution of these derivatives.

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Appendix A

In this appendix we discuss some technical issues regarding the spectrum of the eigenvalue equation for the pricing of asian options.

The general structure of the stationary Schrodinger equation for an Asian Option is of the form

$$\begin{aligned} & (\hat{H} - E)\psi(x) \\ &= -\frac{\sigma^2}{2} \frac{\partial}{\partial x^2} + \left(\frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + (iK e^x - E)\psi = 0 \end{aligned} \quad (\text{A.1})$$

which we rewrite in the form

$$\left(\frac{\partial}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} - \frac{2}{\sigma^2} (iK e^x - E) \right) \psi = 0 \quad (\text{A.2})$$

with $\alpha = (\sigma^2/2 - r)/\sigma^2$, as defined above. The velocity dependent term is eliminated as in the usual BS model by factorizing an overall $e^{\alpha x}$ term in the eigenfunctions.

The eigenfunctions for the Asian Option can be expressed in terms of modified Bessel functions of the first kind $I_\nu(z)$ and $I_{-\nu}(z)$, of complex argument z . Specifically, we introduce the index function

$$v(E) \equiv \frac{\sqrt{\alpha^2 - 2E}}{\sigma} \quad (\text{A.3})$$

then the solution is of the form

$$\begin{aligned} \psi_E(x) = & C_1 e^{\alpha x - i(\pi/4)v(E)} \Gamma(1 - 2v(E)) I_{-2v(E)}(z) \\ & + C_2 e^{\alpha x + i(\pi/4)v(E)} \Gamma(1 + 2v(E)) I_{2v(E)}(z), \end{aligned} \quad (\text{A.4})$$

$$z = |z| e^{i\pi/4}, \quad |z| = 2 \frac{\sqrt{2K}}{\sigma} e^{x/2} \quad (\text{A.5})$$

with C_1 and C_2 arbitrary integration constant. Notice that the eigenfunctions have a branch cut from $-\infty$ to 0.

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