Finite hedging in field theory models of interest rates

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We use path integrals to calculate hedge parameters and efficacy of hedging in a quantum field theory generalization of the Heath, Jarrow, and Morton [Robert Jarrow, David Heath, and Andrew Morton, Econometrica **60**, 77 (1992)] term structure model, which parsimoniously describes the evolution of imperfectly correlated forward rates. We calculate, within the model specification, the effectiveness of hedging over finite periods of time, and obtain the limiting case of instantaneous hedging. We use empirical estimates for the parameters of the model to show that a low-dimensional hedge portfolio is quite effective.

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I. INTRODUCTION

The first interest rate models were spot rate models that had only one factor, which implied that the prices of all bonds[1] were perfectly correlated. This was observed not to be the case in practice, and led Heath and co-workers [1] to develop their famous model [henceforth called the HJM (Heath, Jarrow, and Morton) model]. The most important result of HJM is that, once the discounting factor is fixed, there exists a unique martingale measure for the forward rates. In the HJM model, the forward rate curve can be influenced by more than one factor, and this enabled bond prices to have an imperfect correlation.

However, for a K-factor HJM model, this still meant that the movements in the price of K bonds would determine the movements in the prices of all other bonds. This would enable one to hedge any instrument with K bonds within the framework of this model, which again does not seem to be the case in practice. In fact, if taken to be exact, a two-factor HJM model implies that one can hedge a 30-yr treasury bond with three-month and six-month bills—something that does look not reasonable. Hence, there has been much interest in developing models which do not have this problem. One possibility is to use an infinite-factor HJM model as pointed out by Cohen and Jarrow [2], but it is well known that estimating the parameters of even a two- or three-factor HJM model from market data is very difficult.

These observations led Kennedy [3], Santa-Clara and Sornette [4], and Goldstein [5] to come up with random field models which allowed imperfect correlations across all the bonds. Baaquie [6,7] proceeded with this development by putting all these models into the framework of quantum field theory [8] that allows for the use of a large body of theoretical and computational methods developed in physics to be applied to this problem. The estimation of parameters for different field theory models has been discussed in Baaquie and Srikant [9] and is seen to be more effective than the estimation of parameters in the HJM model.

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[1]In this paper, we only use zero-coupon bonds, hence all references to bonds are to zero-coupon bonds.

II. A SUMMARY OF THE FIELD THEORY MODEL

Let $f(t,x)$ be the forward rates, that is, the interest at time t for an instantaneous loan taken at some time $x>t$ in the future.

We briefly review Baaquie's field theory model of forward rates presented in Refs. [6,7]. Baaquie proposed that the evolution of the forward rates, instead of being driven by white noise processes as is the case for the HJM model, be replaced by considering the forward rates to be a two-dimensional quantum field.

In the K-factor HJM model, the evolution of the forward rates is fixed by

$$\frac{\partial f(t,x)}{\partial t}=\alpha(t,x)+\sum_{i=1}^{K}\sigma_i(t,x)W_i(t), \tag{1}$$

where $W_i(t)$ are Gaussian white noises given by $E[W_i(t)W_j(t')]=\delta_{i-j}\delta(t-t')$.

The main extension that one makes in going over to a quantum field theory is to make the HJM white noise W depend on future time x as well as on t.

Baaquie [6] proposed that the evolution equation for the forward rates to be given by

$$\frac{\partial f(t,x)}{\partial t}=\alpha(t,x)+\sigma(t,x)A(t,x). \tag{2}$$

Both $f(t,x)$ and $A(t,x)$ are two-dimensional quantum fields,[2] and that the field A has the free-(Gaussian) free-field action functional [6]

$$S[A]=-\frac{1}{2}\int_{t_0}^{\infty}dt\int_{t}^{t+T_{\rm FR}}dx\left[A^2+\frac{1}{\mu^2}\left(\frac{\partial A}{\partial x}\right)^2\right], \tag{3}$$

[2]While we can put in many fields A_i, as in the K-factor HJM model, it was shown by Baaquie and Srikant [9] that the extra generality brought into the process due to the extra argument x makes one field sufficient for explaining most of the important features of the market data.

with Neumann boundary conditions imposed at $x=t$ and $x=t+T_{\mathrm{FR}}$. This action has the partition function, obtained by performing the functional integration over the quantum field $A(t,x)$, given by

$$Z[j]=E\left[\exp\left\{\int_{t_0}^{\infty}dt\int_t^{t+T_{\mathrm{FR}}}A(t,x)j(t,x)\right\}\right]$$
$$\equiv\int\mathcal{D}A\exp\left\{\int_{t_0}^{\infty}dt\int_t^{t+T_{\mathrm{FR}}}A(t,x)j(t,x)\right\}e^{S[A]}$$
$$=\exp\left\{\frac{1}{2}\int_{t_0}^{\infty}dt\int_t^{t+T_{\mathrm{FR}}}dx\,dx'\,j(t,x)\times D(x-t,x'-t)j(t,x')\right\} \tag{4}$$

with

$$D(\theta,\theta';T_{\mathrm{FR}})=\mu\frac{\cosh\mu(T_{\mathrm{FR}}-|\theta-\theta'|)+\cosh\mu(T_{\mathrm{FR}}-(\theta+\theta'))}{2\sinh\mu T_{\mathrm{FR}}}$$
$$=D(\theta',\theta;T_{\mathrm{FR}})\quad(\text{symmetric function of }\theta,\theta'), \tag{5}$$

where $\theta=x-t$ and $\theta'=x'-t$. We can calculate expectations and correlations using this partition function. Note that due to the Neumann boundary conditions, the propagator D, in fact, depends only on the difference $x-t$.

When $\mu\to0$, this model should go over to the HJM model, which has been discussed in Refs. [6,9].

The field theory approach preserves the closed form solutions for hedge parameters and futures contracts. Note the (original) finite-factor HJM model cannot accommodate an empirically determined propagator, since it is automatically fixed once the HJM volatility functions are specified.

A detailed empirical study of the field theory model—and the empirical estimation of parameters of the model—was obtained from the forward rate curve by Baaquie and Srikant [9]. The function σ for the Gaussian model has been estimated from market data, and is shown in Fig. 2 in Ref. [9]. The results for the empirical (actual) propagator are found from the data and graphed in Figs. 3 and 4 of Ref. [9]; the implied propagator for the empirically fitted value of $\mu=0.06\ \mathrm{yr}^{-1}$ is shown in Fig. 6 in Ref. [9].

III. HEDGING

All forms of financial instruments are subject to risks due to the unpredictable behavior of the financial markets. There are many ways of defining risk [12]. Hedging is a general term for the procedure of *reducing*, and if possible completely eliminating, the risks to the value of a financial instrument—due to its random fluctuations—by including it in a portfolio together with other related instruments.

For bonds, the main risks are changes in interest rates and the risk of default. In this paper, we are only dealing with default-free bonds so that the only source of risk is the change in interest rates.

The objective in this paper is to investigate how portfolios of bonds behave in field theory models of the interest rates. For the objectives of this paper, we define risk of an instrument to be the standard deviation, or variance, of its final value. This definition of risk is valid for both finite and instantaneous hedgings. Hence, when we hedge a certain instrument, we try to create a portfolio of the hedged and hedging instruments which minimize the overall variance of the portfolio. A perfectly hedged portfolio in this formulation is the one with zero variance.

In the case of a K-factor HJM model, perfect hedging (i.e., a zero variance portfolio) is achievable once any K-independent hedging instrument is used. However, the difficulties introduced by the infinite number of factors in the field theory models have resulted in their being very little literature on this important subject, a notable exception being the measure valued trading strategy developed by Björk, Kabanov, and Runggaldier [13].

We will be primarily concerned with hedging (the fluctuations of) zero-coupon treasury bonds, and we will form hedged portfolios that will include either other bonds with different maturities or futures contracts on bonds. The price of a zero-coupon bond maturing at time T at some time $t<T$ is given by

$$P(t,T)=\exp\left\{-\int_t^T dx\,f(t,x)\right\}. \tag{6}$$

A futures contract on $P(t,T)$ matures at some time $t_F<T$, and its value at some time $t<t_F$ is the futures price $\mathcal{F}(t,t_F,T)$ given by [6]

$$\mathcal{F}(t,t_F,T)=E_{(t,t_F)}[P(t,T)] \tag{7}$$
$$=F(t,t_F,T)\exp\{\Omega_{\mathcal{F}}(t,t_F,T)\}. \tag{8}$$

The forward price is given by

$$F(t,t_F,T)=\frac{P(t,T)}{P(t,t_F)} \tag{9}$$
$$=\exp\left\{-\int_{t_F-t}^{T-t}d\theta\,f(t,\theta)\right\} \tag{10}$$

and the *deterministic* quantity $\Omega_{\mathcal{F}}(t,t_F,T)$ is given by [6]

$$\Omega_{\mathcal{F}}(t_0,t_F,T)=-\sum_{i=1}^{N}\int_{t_0}^{t_F}dt\int_0^{t_F-t}d\theta\,\sigma_i(t,\theta)\times\int_{t_F-t}^{T-t}d\theta'\,\sigma_i(t,\theta'). \tag{11}$$

A typical hedged portfolio that is formed out of bonds with varying maturities T_i is given by

$$\Pi(t)=P(t,T)+\sum_{i=1}^{N}\Delta_i P(t,T_i), \tag{12}$$

whereas a hedged portfolio using futures contracts has the form

$$\Pi(t)=P(t,T)+\sum_{i=1}^{N}\Delta_i\mathcal{F}(t,t_F,T_i). \tag{13}$$

In this paper the weights Δ_i of the hedged portfolio will be determined from the field theory model for the forward rates.

Instantaneous hedging refers to a process where the portfolio Π is continuously rebalanced. In Ref. [15] we carried out a detailed analysis of the instantaneous hedging of a bond based on the field theory model, and to do so one requires only the propagator and the evolution equations for the forward rates. In constrast, we will see that for the case of finite hegding the detailed structure of the path integral becomes important for the derivations.

In practice, continuous hedging is not carried out due to transaction costs. We hence consider finite time hedging since it is important in practice. Hedging over a finite time horizon t_* means creating a portfolio at t, namely, $\Pi(t)$, and then letting this portfolio evolve over the time interval $[t,t_*]$ without any further rebalancing. We will take the limit of infinitesimal time and recover the results of instantaneous hedging from the finite case.

Finite time hedging provides a measure on how frequently the portfolio needs to be rebalanced, and hence provides a way of optimizing between gains obtained through hedging against expenses incurred due to transaction costs.

IV. FINITE TIME HEDGING

We only consider the hedging of bonds with other bonds as the calculations for minimizing variance can be done exactly. We will not do hedging of bonds with futures—even though this can also be solved exactly by minimizing the variance—as it does not add much extra insight for finite time. To see this, consider hedging with a futures contract on a zero-coupon bond of duration T that matures at the same as the hedging horizon. This gives exactly the same result as that obtained by hedging with a bond of the same maturity T. Therefore, we gain nothing by carrying out that calculation.

We fix some notation. Let us denote the initial time by t_0, the hedging horizon by t_*, and the maturities of the bonds used for hedging by T_i. We would like to create a portfolio today to hedge a treasury bond, say $P(t_0,T)$.

We consider the hedging of one bond maturing at T with N other bonds maturing at T_i, $1\leqslant i\leqslant N$. If one of the $T_i=T$, then the solution is trivial since it is the same bond. The hedging is then just to short (sell) the same bond giving us a zero portfolio with obviously zero variance. Since this solution is uninteresting, we assume that $T_i\neq T\ \forall\ i$.

Recall from Eq. (12) that a hedged portfolio is given by

$$\Pi(t_0)=P(t_0,T)+\sum_{i=1}^{N}\Delta_i P(t_0,T_i). \tag{14}$$

Note that the portfolio $\Pi(t)$, for $t>t_0$, is not a log normal (Guassian) random variable; however, we continue to consider its variance to be a suitable measure of the fluctuations in its value. Hence, the weights of the bonds $P(t_0,T_i)$ with maturities T_i, namely, Δ_i, are chosen so that the variance of the portfolio $\Pi(t_*)$ (at future time t_*) is a minimum. We hence need to compute the variance

$$\begin{aligned}\mathrm{var}[\Pi(t_*)]&\equiv E[\Pi^2(t_*)]-\{E[\Pi(t_*)]\}^2\\&=E[P^2(t_*,T)]+2\sum_{i=1}^{N}\Delta_i E[P(t_*,T)P(t_*,T_i)]\end{aligned} \tag{15}$$

$$\begin{aligned}&+\sum_{i,j=1}^{N}\Delta_i\Delta_j E[P(t_*,T_i)P(t_*,T_j)]\\&-\{E[\Pi(t_*)]\}^2.\end{aligned} \tag{16}$$

The coefficients Δ_i are fixed by mininimizing $\mathrm{var}[\Pi(t_*)]$.

To be able to optimally hedge a bond $P(t_0,T)$ with other bonds (in the sense of having a minimal resulting variance) we need to evaluate the covariance between the values of bonds of different maturities at time $t=t_*$. Since the initial conditions are given at $t=t_0$, we make the following simplifications. Making use of Eq. (9), we have

$$P(t_*,T_i)\equiv F(t_0,t_*,T)e^{-G_i}, \tag{17}$$

$$G_i=\int_{t_*}^{T_i}dx[f(t_*,x)-f(t_0,x)]. \tag{18}$$

In other words,

$$G_i=-\ln\left(\frac{P(t_*,T)}{F(t_0,t_*,T_i)}\right)=-\ln\left(\frac{P(t_*,T_i)P(t_0,t_*)}{P(t_0,T_i)}\right). \tag{19}$$

Note that the forward rate $F(t_0,t_*,T_i)$ is an initial condition that is fixed by market data at $t=t_0$.

A typical correlator of bonds can be written as

$$E[P(t_*,T_i)]=F(t_0,t_*,T_i)E[e^{-G_i}], \tag{20}$$

where

$$\begin{aligned}E[e^{-G_i}]&\\&=\int_{-\infty}^{+\infty}dG_i\,e^{-G_i}E\left[\delta\left\{\int_{t_*}^{T_i}dx[f(t_*,x)-f(t_0,x)]-G_i\right\}\right]\\&=\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\int dG_i\,e^{-G_i}E\left[\exp ip\left\{\int_{t_*}^{T_i}dx[f(t_*,x)\right.\right.\\&\left.\left.\quad-f(t_0,x)]-G_i\right\}\right].\end{aligned} \tag{21}$$

In general, to calculate the covariance between bonds of varying maturities, we first find the joint probability density function for N bonds at the hedging horizon. We calculate the joint distribution of the quantities which represent the logarithms of the ratios of the final value of the bonds to the value at the initial time. The following calculation proceeds efficiently because of the use of path integral techniques, which are very useful for such problems.

Consider

$$E\left[\prod_{j=1}^{N}\delta\left\{\int_{t_*}^{T_j}dx[f(t_*,x)-f(t_0,x)]-G_j\right\}\right]$$
$$=\prod_{j=1}^{N}\int\frac{dp_j}{2\pi}\int\mathcal{D}A\,e^{S[A]}$$
$$\times\exp\left\{i\sum_{j=1}^{N}p_j\left(\int_{t_0}^{t_*}dt\int_{t_*}^{T_j}dx\,\alpha(t,x)+\int_{t_0}^{t_*}dt\int_{t_*}^{T_j}dx\,\sigma(t,x)A(t,x)-G_j\right)\right\},\qquad(22)$$

which, on applying Eq. (4), becomes

$$\prod_{j=1}^{N}\int\frac{dp_j}{2\pi}\exp\left\{-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}p_jp_k\times\int_0^{t_*}dt\int_{t_*}^{T_j}dx\int_{t_*}^{T_h}dx'\,\sigma(t,x)D(x-t,x'-t)\sigma(t,x')+i\sum_{j=1}^{N}p_j\left(\int_0^{t_*}dt\int_{t_*}^{T_j}dx\,\alpha(t,x)-G_j\right)\right\}.\qquad(23)$$

Performing the Gaussian integrations, we obtain the joint probability distribution given by

$$(2\pi)^{-N/2}(\det B)^{-1/2}\exp\left\{-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}(G_j-m_j)\times B_{jk}^{-1}(G_k-m_k)\right\},\qquad(24)$$

where B is the matrix whose elements B_{ij} are given by

$$B_{ij}=\int_{t_0}^{t_*}dt\int_{t_*}^{T_i}dx\int_{t_*}^{T_j}dx'\,\sigma(t,x)D(x-t,x'-t)\sigma(t,x')\qquad(25)$$

and m_i is given by

$$m_i=\int_{t_0}^{t_*}dt\int_{t_*}^{T_i}dx\,\alpha(t,x).\qquad(26)$$

Hence, the quantities G_i follow a multivariate Gaussian distribution with covariance matrix B_{ij} and mean m_i. Define

$$\int DG\equiv(2\pi)^{-N/2}(\det B)^{-1/2}\prod_{j=1}^{N}\int_{-\infty}^{+\infty}dG_j,\qquad(27)$$

$$S[G]\equiv-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}(G_j-m_j)B_{jk}^{-1}(G_k-m_k).\qquad(28)$$

Having obtained the joint distribution of G_i, we can find the covariance of the final bond prices by tabulating the expectations of each of the bonds and the expectation of their products. The final bond price in terms of G_i is given by $P(t_*,T_i)=F(t_0,t_*,T_i)e^{-G_i}$. Hence, the expectation of this quantity is given by

$$E[P(t_*,T_i)]=F(t_0,t_*,T_i)\int DG\,e^{-G_i}e^{S[G]}=\mathcal{F}(t_0,t_*,T_i),\qquad(29)$$

as expected, since the expectation of the future bond price is the future's price.

The expectation of the products of the prices of two bonds is given by

$$E[P(t_*,T_i)P(t_*,T_j)]=F(t_0,t_*,T_i)F(t_0,t_*,T_j)\times\int DG\,e^{-G_i-G_j}e^{S[G]}.\qquad(30)$$

On evaluation, this gives the result

$$E[P(t_*,T_i)P(t_*,T_j)]=\mathcal{F}(t_0,t_*,T_i)\mathcal{F}(t_0,t_*,T_j)\exp\left\{\int_{t_0}^{t_*}dt\int_{t_*}^{T_i}dx\times\int_{t_*}^{T_j}dx'\,\sigma(t,x)D(x-t,x'-t)\sigma(t,x')\right\}.\qquad(31)$$

The covariance between the prices $P(t_*,T_i)$ and $P(t_*,T_j)$ is given by

$$M_{ij}=E[P(t_*,T_i)P(t_*,T_j)]-E[P(t_*,T_i)]E[P(t_*,T_j)],\qquad(32)$$

and hence

$$M_{ij}=\mathcal{F}(t_0,t_*,T_i)\mathcal{F}(t_0,t_*,T_j)\times\left\{\exp\left(\int_0^{t_*}dt\int_{t_*}^{T_i}dx\int_{t_*}^{T_j}dx'\,\sigma(t,x)\times D(x-t,x'-t)\sigma(t,x')\right)-1\right\},\qquad(33)$$

and the covariance between the hedged bond of maturity T and the hedging bonds of maturity T_i is given by

$$L_i=\mathcal{F}(t_0,t_*,T)\mathcal{F}(t_0,t_*,T_i)\times\left\{\exp\left(\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\int_{t_*}^{T_i}dx'\,\sigma(t,x)\times D(x-t,x'-t)\sigma(t,x')\right)-1\right\}.\qquad(34)$$

Minimization of the residual variance of the hedged portfolio is straightforward, and the hedge ratios are found to be given by

$$\Delta_i = -\sum_{j=1}^{N} L_j M_{ji}^{-1}. \tag{35}$$

We hence have the hedged portfolio given by

$$\Pi(t_0) = P(t_0,T) + \sum_{i=1}^{N} \Delta_i P(t_0,T_i) \tag{36}$$

with the portfolio's minimized *residual variance* being given by

$$\text{var}[\Pi(t_*)] = \text{var}[P(t_*,T)] - L^T M^{-1} L. \tag{37}$$

The residual variance enables the effectiveness of the hedged portfolio to be evaluated. In the following section, residual variance is used for studying the hedged portfolios that include bonds of different maturities.

One important difference between instantaneous hedging and finite time hedging is that in the latter case the result depends on the value of the drift velocity α. For finite time hedging, it is natural that α should appear. The reason is that if one is not hedging continuously, then the portfolio is exposed to market risks, and therefore risk premiums defined in terms of α appear in the formulas for finite time hedging.

In the calculation above we used the risk-neutral drift α, obtained by using the money market as the (discounting) numeraire. However, the market does not follow the risk-neutral measure and it would be better to use a value for α estimated from the market for any practical use of this method. For the case of instantaneous hedging, the difference between the risk-neutral and market drift is irrelevant, since in the very short term only the stochastic term dominates, making the drift term itself inconsequential. This, of course, is not the case for the finite time case where the drift becomes important (it is not difficult to see that the importance of the drift grows with the time horizon).

V. SEMIEMPIRICAL RESULTS FOR FINITE TIME HEDGING

We discuss the empirical results for hedging of a bond with other bonds for both the best fit for the constant rigidity field theory model as well as for the fully empirical propagator. Reduction of residual variance to zero is not feasible in practice; the best one can do is to decide the level of risk one is prepared to live with, and then include as many hedging instruments as is required to achieve this level of risk.

We take the current forward rate curve to be flat and equal to 5% throughout. The initial forward rate curve does not affect any of the qualitative results. The results can also be easily extended to other bonds.

The calculation of L and M were carried out using simple trapezoidal integration as the data is not exceptionally accurate in the first place. Volatility σ was assumed to be purely a function of $\theta = x - t$ so that all the integrals over x were replaced by integrals over θ. The bond to be hedged was chosen to be the 5-yr-zero-coupon bond and the time horizon t_* was chosen to be one year.

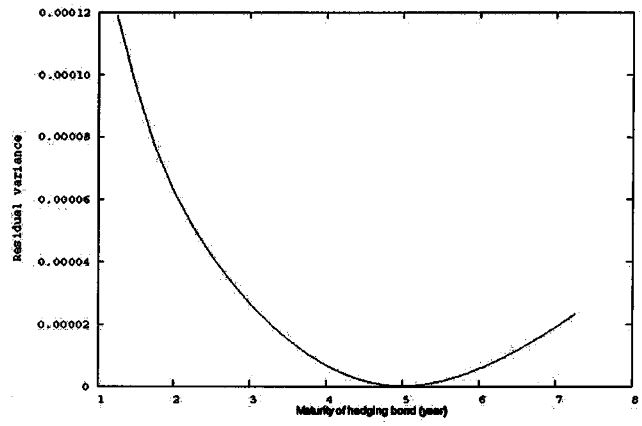

FIG. 1. Residual variance when a 5-yr bond is hedged with one other bond, with the best fit of the constant rigidity field theory model. Time horizon for hedging of one year. Residual variance $= \text{var}[P(1\text{ yr},5\text{ yr})] - L_1^2/M_{11}$.

Note that the errors involved largely cancel themselves out, and hence the residual variances obtained are still quite accurate. The parabolic nature of the residual variance as shown in Fig. 1 is because μ is constant; this graph appeals to our economic intuition which suggests that the correlation between forward rates decreases monotonically as the distance between them increases.

A more complicated dependence on μ and maturity would produce residual variances that do not deviate monotonically as the maturities of the underlying bonds and the hedge portfolio increase.

The residual variance and hedge ratio of the hedged portfolio for the hedging a bond using another bond—using the constant rigidity field theory model—is shown in Figs. 1 and 2. The residual variance of the hedged portfolio using two bonds for hedging is shown in Fig. 3.

The results for the hedging of one bond using the empiri-

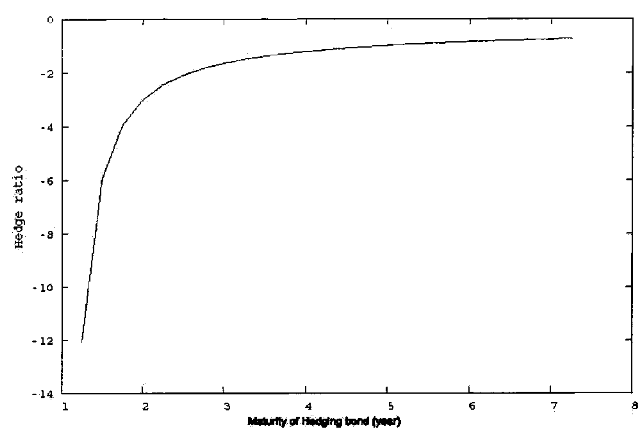

FIG. 2. Hedge ratio when a 5-yr bond is hedged with another bond with the best fit of the constant rigidity field theory model. Time horizon of hedging of one year. Hedge ratio $\Delta_1 = -L_1^2/M_{11}$.

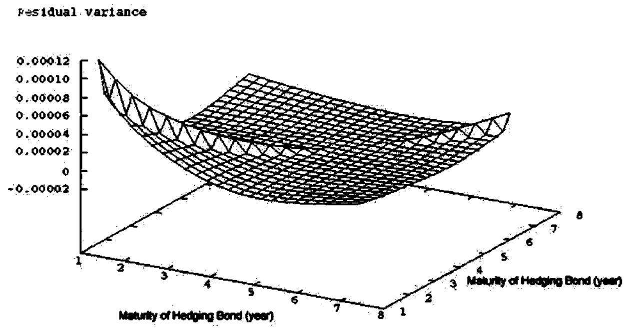

FIG. 3. Residual variance when a 5-yr bond is hedged with two other bonds with the best fit of the constant rigidity field theory model. Time horizon of hedging of one year.

cal propagator for the hedged portfolio, namely, its residual variance and hedge ratio gives results almost identical to the one obtained using the field theory propagator with the best fit for the rigidity parameter $\mu = 0.06\ \mathrm{yr}^{-1}$. As is to be expected, the empirical rather than the field theory propagator gives a better hedged portfolio.

Note the residual variance when two bonds are used to form the hedged portfolio has instabilities when the maturity of the two bonds being used for hedging have nearby maturities, and is an important result that also emerges for the case of instantaneous hedging [15]. The field theory model shows that if one was to form the Greeks for this hedged portfolio, the instabilities that have surfaced in the field theory model would lead to large coefficients, and could be of some significance in choosing the optimum maturity for the bonds being used for hedging.

One interesting result of finite time hedging is that the actual residual variance of the hedged portfolio when hedging over a finite time horizon is less than what one naively extrapolates the infinitesimal hedging result. This seems to be due to the fact that the domain of the forward rates that contribute to the variance of the bonds reduces as the time horizon increases. This is very clear if the maturity of the bond is close to the hedging horizon, since the volatility of bonds reduces quickly as the time to maturity approaches. Apart from this reduction, the results look very similar to the infinitesimal case. This is probably due to the fact that the volatility is quite small so the nonlinear effects in the covariance matrix M_{ij} given in Eq. (33) are not apparent.

If very long time horizons (ten years or more) and long term bonds are considered, the results will probably be quite different.

VI. INSTANTANEOUS HEDGING

In instantaneous hedging, we consider a hedging portfolio which is rebalanced continuously in time. Hence, we are only concerned with the instantaneous variance of the portfolio. To find the weights of the hedged portfolio, we minimize the variance of $d\Pi(t_0)/dt$. To obtain the results for instantaneous hedging, note that for $t_* = t_0 + \epsilon$ we have

$$\mathrm{var}\left[\frac{d\Pi(t_0)}{dt}\right] \simeq \frac{1}{\epsilon^2}\,\mathrm{var}[\Pi(t_0+\epsilon)], \tag{38}$$

since the value of $\Pi(t_0)$ is deterministic. Both M_{ij} and L_i computed for finite time hedging yield—after appropriate scaling by ϵ—a finite limit on taking $\epsilon \to 0$.

We summarize in Table I the results for instantaneous hedging both for a hedged portfolio composed out of zero-coupon bonds, and out of futures contracts. We use the notation $P(t_0,T_i) \equiv P_i$ and $\mathcal{F}(t_0,t_F,T_i) \equiv \mathcal{F}_i$. The result for instantaneous hedging for a portfolio composed out of bonds with varying maturities follows directly from taking the limit of the finite hedging case.

A detailed analysis of instantaneous hedging using the field theory model for the forward rates is given in Ref. [15].

TABLE I. Residual variance and hedging weights for hedged portfolios of a zero coupon bond for instantaneous hedging using other bonds and future contracts.

Portfolio II	Residual variance of portfolio $V = \mathrm{Var}\left[\frac{d\Pi(t_0)}{dt}\right]$	Weights Δ_i
P	$V_0 = P^2 \int_0^{T-t_0} d\theta \int_0^{T-t_0} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})$	0
$P + \sum_{i=1}^{N} \Delta_i P_i$	$V = V_0 - L^T M^{-1} L$ $L_i = P P_i \int_0^{T-t_0} d\theta \int_0^{T_i-t_0} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})$ $M_{ij} = P_i P_j \int_0^{T_i-t_0} d\theta \int_0^{T_j-t_0} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})$	$-\sum_{j=1}^{N} L_j M_{ji}^{-1}$
$P + \sum_{i=1}^{N} \Delta_i \mathcal{F}_i$	$V = V_0 - L^T M^{-1} L$ $L_i = P\mathcal{F}_i \int_{t_F-t_0}^{T_i-t_0} d\theta \int_0^{T-t_0} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta')$ $M_{ij} = \mathcal{F}_i \mathcal{F}_j \int_{t_F-t_0}^{T_i-t_0} d\theta \int_{t_F-t_0}^{T_j-t_0} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta')$	$-\sum_{j=1}^{N} L_j M_{ji}^{-1}$

The most important result is that one achieves a large reduction in the residual variance by shortening two futures contracts that mature before and after the maturity of the bond being hedged. If one includes three or more futures contracts, there is relatively negligible gains in the residual variance.

VII. CONCLUSION

We have shown that the field theory model offers techniques to calculate hedge parameters for fixed income derivatives and provides a framework to answer questions concerning the number and maturity of bonds to include in a hedge portfolio. We have also seen how the field theory model can be used to estimate hedge parameters for both the finite time as well for the instantaneous case. We have used the field theory model calibrated to market data to show that a low-dimensional basis provides a reasonably good approximation within the framework of this model.

The results of this analysis show that field theory models effectively address the theoretical dilemmas of finite-factor term structure models, and offer a practical alternative to finite-factor models.

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